SOLUTION OF THE PROBLEM OF CONVECTIVE HEAT EXCHANGE DURING THE TURBULENT FLOW OF A LIQUID IN A PLANE-PARALLEL CHANNEL USING THE PAI VELOCITY PROFILE

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UDC 536.411:532.517.4

A solution is obtained for the problem of convective heat exchange in a plane-parallel channel with turbulent flow of the liquid and an arbitrary ratio of heat fluxes at the walls.

According to Pai, the distribution of the velocity w and the coefficient of turbulent kinematic viscosity ε_t over the cross section of a cylindrical or plane-parallel channel (Fig. 1) is determined by the equations

$$\Pi_w(\eta) = \frac{w}{w_{max}} = 1 - \frac{n-s}{n-1}\eta^2 - \frac{s-1}{n-1}\eta^{2n}, \tag{1}$$

$$\Pi_\varepsilon(\eta) = 1 + \frac{\varepsilon_t}{\nu} = \frac{s(n-1)}{n-s+n(s-1)\eta^{2n-2}}. \tag{2}$$

On the basis of an analysis of the data of [1], the values of n and s for a plane-parallel channel in the region of log Re_e = 4-6 can be determined from the following functions (with a deviation of less than 1% from the data of [1]):

$$\begin{gathered} s = 0{,}00275\,\mathrm{Re}_e^{0{,}8} = 10\text{—}200, \\ n = 1{,}5s. \end{gathered} \tag{3}$$

Using Eqs. (3), we obtain

$$\Pi_w(\eta) = 1 - \frac{s}{3s-2}\eta^2 - \frac{2(s-1)}{3s-2}\eta^{3s}, \tag{4}$$

$$\Pi_\varepsilon(\eta) = \frac{3s-2}{1+3(s-1)\eta^{3s-2}}. \tag{5}$$

Let us consider the case of heat exchange in a plane-parallel channel in the section of hydrodynamically and thermally stabilized turbulent flow of a liquid and with an arbitrary ratio of specific heat fluxes q_1 and q_2, constant in value, supplied by the liquid to the lower and upper walls 1 and 2.

We will take the physical properties of the liquid as independent of the temperature, the liquid is incompressible, and we neglect heat transfer along the channel axis and the energy of dissipation. In this case the energy equation has the form

$$\rho c_p w\frac{\partial T}{\partial x} - \frac{\partial}{\partial y}\left[(\lambda + \lambda_t)\frac{\partial T}{\partial y}\right] = 0. \tag{6}$$

For convective heat exchange with a fully developed temperature profile and a constant heat flux density at the walls the following equation [3] is valid:

$$\frac{\partial T}{\partial x} = \frac{dT}{dx} = \frac{d\bar{T}}{dx} = \text{const}. \tag{7}$$

Moscow Institute of Railroad Transport Engineers. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 2, pp. 261-266, August, 1975. Original article submitted November 5, 1973.

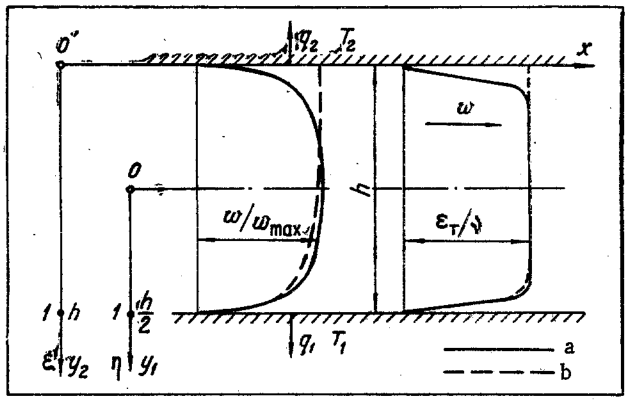

Fig. 1. Diagram of channel: a) profiles W/W_{max} and ε_t/ν according to Eqs. (4) and (5); b) the same according to Eqs. (18) and (19).

From the thermal balance for an element of liquid [3], with allowance for (7), one can obtain

$$\rho c_p \bar{w} \frac{d\bar{T}}{dx} = -\frac{q_1 + q_2}{h}. \tag{8}$$

We can write $\bar{w}$ as

$$\bar{w} = \frac{\bar{w}}{w_{max}} \cdot \frac{w_{max}}{w} w = \frac{w k_w}{\Pi_w},$$

where

$$k_w = \frac{\bar{w}}{w_{max}} = \int_0^1 \Pi_w(\eta)\, d\eta$$

is a constant value for a given Re_e.

Let us consider the case of symmetrical heat removal ($q_1 = q_2$) and the case with one adiabatic wall ($q_2 = 0$), assigning to the values the subscripts s and a, respectively. We introduce the dimensionless temperature

$$\theta = \frac{T - T_{wa.1}}{\bar{T} - T_{wa.1}}. \tag{9}$$

Since the Nusselt number is

$$Nu = \frac{2hq_1}{\lambda(\bar{T} - T_{wa.})},$$

we can write the energy equation (6) for the case of $q_1 = q_2$ in dimensionless coordinates:

$$\frac{Nu_s \Pi_w(\eta)}{n_s k_s} + \frac{d}{d\eta}\left[\Pi_a(\eta) \frac{d\theta_s}{d\eta}\right] = 0, \tag{10}$$

where the coefficient $n_s = 4$ and

$$\Pi_a(\eta) = 1 + \frac{a_t}{a} = \frac{Pr}{Pr_t}\left(\frac{Pr_t}{Pr} + \frac{\varepsilon_t}{\nu}\right). \tag{11}$$

We will solve Eq. (10) with the boundary conditions

$$\theta'(0) = 0; \quad \theta(1) = 0. \tag{12}$$

As a result of the solution constructed by the method presented in [1], we obtain the general expressions for Nu_s and θ_s:

$$\frac{1}{Nu_s} = \frac{1}{n_s k_s^2} \int_0^1 d\eta \int_\eta^1 \frac{\int_0^\eta \Pi_w(\eta)\, d\eta}{\Pi_a(\eta)}\, d\eta, \tag{13}$$

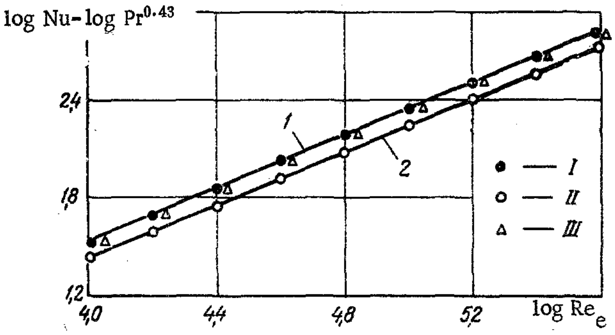

Fig. 2. Functions $Nu = f(Re_e, Pr)$ with $Pr = 0.6$ and $Pr_t = 0.8$: 1, 2) from Eqs. (17) and (22); I, II, III) from Eqs. (15), (20), and (24), respectively.

$$\theta_s = \frac{Nu_s}{n_s k_s} \int_\eta^1 \frac{\int_0^\eta \Pi_w(\eta)\, d\eta}{\Pi_a(\eta)}\, d\eta. \tag{14}$$

The analytical integration of Eqs. (13) and (14) is possible when $Pr = Pr_t$. In this case $\Pi_\alpha(\eta) = \Pi_\varepsilon(\eta)$.

One can also obtain an approximate analytical solution with numbers $Pr = 0.55$-0.7, which are characteristic for gaseous media, and numbers $Pr_t = 0.9$-1, which are usually adopted in heat-exchange calculations.

With these values of Pr and Pr_t, assuming that $\varepsilon_t/\nu >> 1$ over the entire channel cross section except for the region directly adjacent to the wall, one can assume in Eq. (11) that

$$\frac{Pr_t}{Pr} + \frac{\varepsilon_t}{\nu} \approx 1 + \frac{\varepsilon_t}{\nu} = \Pi_\varepsilon(\eta).$$

As a result of integration, after transformations and slight approximating simplifications (in operations with high powers) which do not reduce the accuracy, we obtain

$$Nu_s = \frac{n_s k_{ws}^2 k_1 Pr}{k_s Pr_т}; \tag{15}$$

$$\theta_s = \frac{k_{ws}}{k_s}\left[k_{6s} - (0{,}5\eta^2 - k_{4s}\eta^4 + k_{5s}\eta^{3s})\right]; \tag{16}$$

$$\theta_{s,max} = \frac{T - T_{wa.1}}{T_{max} - T_{wa.1}} = \frac{k_{6s} - (0{,}5\eta^2 - k_{4s}\eta^4 + k_{5s}\eta^{3s})}{k_{6s}}, \tag{16a}$$

where

$$k_1 = 3s - 2; \quad k_2 = \frac{s}{k_1}; \quad k_3 = 1 - k_2; \quad k_{4s} = \frac{k_2}{12};$$

$$k_{5s} = \frac{2k_3(2s^2 - 1)}{s(3s + 2)}; \quad k_{6s} = 0{,}5 - k_{4s} + k_{5s};$$

$$k_{ws} = \frac{8s}{3(3s + 1)}; \quad k_s = k_{6s} - \frac{0{,}5 + k_{6s}k_2}{3} + \frac{7k_{4s}}{5} -$$

$$- \frac{k_{5s} + k_3 k_{6s}}{3s + 1} - \frac{k_2 k_{4s}}{7} + \frac{k_3 k_{5s}}{6s + 1} - \frac{k_{4s} k_3}{3s + 5} + \frac{k_2 k_{5s} + 0{,}5k_3}{3s + 3}.$$

With $Pr = 0.6$ and $Pr_t = 0.8$ the calculated data are well approximated by an equation which is standard for gases (Fig. 2):

$$Nu_s = 0.022 Re_e^{0.8} Pr^{0.43}. \tag{17}$$

In the case of $q_2 = 0$ the energy equation, the boundary conditions, and the solution have the same form as Eqs. (10), (12)-(14). It is only necessary to replace the coordinate η by ξ, the subscript "s" by "α," and take $n_\alpha = 2$. The direct use of the Pai profile here is not possible, however, since it was required to calculate $(-\xi)^k$, where k can be any value, even fractional.

The form of the w and ε_t profiles has the strongest effect in the zone of intense heat exchange, i.e., near the lower wall. The nature of the profile near the upper adiabatic wall does not play an important role. Let us take the form of the w and ε_t profiles such that they coincide with the Pai profiles in the region of the lower wall, placing the origin 0' of the coordinates at the upper wall (Fig. 1). Then the values of w and ε_t will be maximal at the point 0':

$$\Pi_w(\xi) = 1 - \frac{s}{3s - 2}\xi^4 - \frac{2(s - 1)}{3s - 2}\xi^{6s}, \tag{18}$$

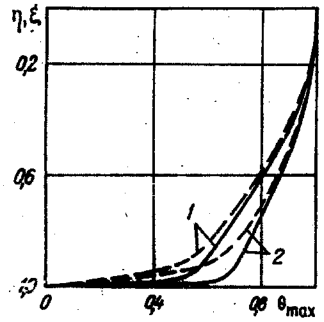

Fig. 3. Dimensionless temperature profiles: 1) adiabatic wall; 2) symmetrical heat removal (dashed curve: $Re_e = 10^4$, s = 4; solid curve: $Re_e = 10^5$, s = 26).

$$\Pi_\varepsilon(\xi) = \frac{3s-2}{1+3(s-1)\xi^{6s-4}}. \tag{19}$$

We can test the correctness of this method further by obtaining the solution with symmetrical heat removal from the solution for an adiabatic wall by the method of superposition [2, 3].

With the use of (18) and (19) the solution for an adiabatic wall takes the form

$$\mathrm{Nu_a} = \frac{n_a k_{wa}^2 k_1 \mathrm{Pr}}{k_a \mathrm{Pr_t}}; \tag{20}$$

$$\theta_a = \frac{k_{wa}}{k_a}[k_{6a} - (0.5\xi^2 - k_{4a}\xi^6 + k_{5a}\xi^{6s})]; \tag{21}$$

$$\theta_{a,\max} = \frac{k_{6a} - (0.5\xi^2 - k_{4a}\xi^6 + k_{5a}\xi^{6s})}{k_{6a}}, \tag{22}$$

where

$$n_a = 2;\ k_{wa} = \frac{5.6s}{6s+1};\quad k_{4a} = \frac{s}{30k_1};\quad k_{5a} = \frac{(s-1)(7s-5)}{5sk_1};$$

$$k_{6a} = 0.5 - k_{4a} + k_{5a};\quad k_a = k_{6a} - \frac{1}{6} - \frac{k_2 k_{6a}}{5} + \frac{16k_{4a}}{7} -$$

$$- \frac{k_{5a}}{6s-1} - \frac{k_3 k_{6a}}{6s-1} - \frac{k_2 k_{4a}}{11} + \frac{k_2 k_{5a} + 0.5k_3}{6s+3} - \frac{k_3 k_{4a}}{6s+7} + \frac{k_3 k_{5a}}{12s-1}.$$

With Pr = 0.6 and $\mathrm{Pr_t}$ = 0.8 the approximation of the results of the calculation (Fig. 2) gives

$$\mathrm{Nu_1} = \frac{\mathrm{Nu_a}}{1-(q_2/q_1)\theta^*}. \tag{23}$$

When Pr = $\mathrm{Pr_t}$ the approximate equations (15), (16) and (20), (21) become exact.

From Eq. (20) for the case of q_2 = 0 one can obtain an expression for Nu with arbitrary values of q_1 and q_2 by using an equation obtained in [2] by the method of superposition:

$$\theta^* = \frac{T_{2a} - \overline{T}}{\overline{T} - T_{wa,1}} = \theta_a(0) - 1, \tag{24}$$

in which

$$\theta_a(0) = \frac{k_{wa}k_{6a}}{k_a}. \tag{25}$$

According to the results of the calculation the value θ* equals 0.204, 0.194, and 0.189 when $Re_e = 10^4$, 10^5, and 10^6, respectively.

In the particular case when $q_2 = q_1$ we obtain from (20) and (24) an approximate expression for $\mathrm{Nu_s}$:

$$\mathrm{Nu_s} \cong \frac{\mathrm{Nu_a}}{1-\theta^*}. \tag{26}$$

A comparison of the results of the calculation by the exact equation (15) and the approximate equation (26) is shown in Fig. 2, from which it is seen that the agreement of the results is good, which confirms the correctness of the modification of the w and ε_t profiles performed above.

To complete the analysis the dimensionless temperature profiles

$$\theta_{\max} = \frac{T - T_{wa,1}}{T_{\max} - T_{wa,1}} = \frac{\theta}{\theta(0)}$$

for the cases of $q_1 = q_2$ and $q_2 = 0$ are shown in Fig. 3. As seen from Figs. 2 and 3, in the channel with an adiabatic wall the temperature profile is found to be less full and the Nusselt number is 25% lower than in the case of symmetrical heat removal.

NOTATION

T, T_{wa}, temperatures of liquid and wall; $\eta = y_1/0.5h_1$, $\xi = y_2/h$, dimensionless coordinates; ν, ρ, λ, a, c_p, kinematic viscosity, density, thermal conductivity, and thermal diffusivity of liquid and its heat capacity at constant pressure; a_t, $\lambda_t = a_t \rho c_p$, coefficients of turbulent heat transfer and of turbulent thermal conductivity; $Pr = \nu/a$, $Pr_t = \varepsilon_t/a_t$, Prandtl number and turbulent Prandtl number; $Re_e = wd_e/\nu$, Reynolds number; $d_e = 2h$, equivalent diameter of channel; n, s, coefficients in Pai's equations, dependent on Re. Indices: max, 1, and 2 denote the largest value in the cross section and values pertaining to the lower and upper walls; a bar above a value denotes its average over the cross section.

LITERATURE CITED

1. L. V. Baldwin and R. D. Haberstroh, Am. Inst. Chem. Eng. J., 14, No. 5, 825 (1968).
2. B. S. Petukhov and L. I. Roizen, Inzh.-Fiz. Zh., 6, No. 3 (1963).
3. W. M. Kays, Convective Heat and Mass Transfer, McGraw-Hill, New York (1966).
4. S. I. Pai, J. Appl. Mech., 20, 109 (1953).
5. S. I. Pai, J. Franklin Inst., 256, 337 (1953).